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Exclusion Process and Droplet Shape

J.-P. Marchand[1] **and Ph. A. Martin**[2]

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We use a mathematical isomorphism between the one-dimensional exclusion process and the two-dimensional stochastic Ising model in the low-temperature limit to describe the typical instantaneous shape of a supercritical droplet. We derive, specifically, the exact asymptotic distribution of the boundaries of a $(+1)$ spin region, confined to Z_+^2 and subjected to a positive magnetic field. In an appropriate scaling, the boundary distribution converges to a deterministic continuum limit.

KEY WORDS: Exclusion process; Ising models; Glauber dynamics; droplet shape.

1. INTRODUCTION

Let η_t be the one-dimensional asymmetric exclusion process in $X = \{0, 1\}^Z$ corresponding to the semigroup S_t (Feller process) with generator

$$(Gf)(\eta) = \sum_{\substack{\eta(k)=1 \\ \eta(l)=0}} p(k, l)[f(\eta_{kl}) - f(\eta)]$$

where η_{kl}, $k, l \in Z$, is the configuration obtained from η by exchanging k and l.[1] The transition probabilities are defined by

$$p(k, k+1) = p, \qquad p(k, k-1) = q, \qquad p + q = 1$$

and $p(k, l) = 0$ otherwise.

[1] University of Denver, Denver, Colorado. On leave at the EPFL, CH-1015 Lausanne, Switzerland.
[2] École Polytechnique Fédérale de Lausanne, Switzerland.

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We assume that the probability measure (state) ν on X is initially concentrated on the configuration

$$\eta_0(k)=\begin{cases}1 & k\leqslant 0\\ 0 & k>0\end{cases}$$

and consider its evolution $\nu_t = S_t \nu_0$ in time.

The system behaves differently for $r = p/q$ larger or smaller than one. If $r > 1$, the particles diffuse to the right with a drift linear in time, and the system does not approach any stationary state starting from η_0. (There is, however, a well-defined asymptotic distribution if the process is viewed from a frame in uniform translation.[2]) If $r < 1$, ν_t converges weakly to an invariant product state.[3]

$$\nu_\infty(\eta) = \prod_{l:\eta(l)=1} \rho(l) \cdot \prod_{l:\eta(l)=0} (1-\rho(l)) \tag{1.1}$$

with density

$$\rho(l) = \nu_\infty\{\eta : \eta(l) = 1\} = r^l(1 + r^l)^{-1} \tag{1.2}$$

The model has two interesting physical interpretations. The second will be the main object of our study.

(i) Particle Motion in an External Field

The process describes the random walk (with exclusion) on Z of charged particles in a constant electric field E. With Hamiltonian $H(\eta) = -E\sum_k k\eta(k)$ and transition probabilities defined by the detailed balance condition $p(k, l)\exp(-\beta H(\eta)) = \exp(-\beta H(\eta_{kl}))\, p(l, k)$, one finds

$$r = \frac{p}{q} = e^{\beta E}$$

A positive field $E > 0$ ($r > 1$) generates a drift to the right. For $E < 0$, the stationary state ν_∞ is the Gibbs state with respect to the Hamiltonian $H(\eta)$ and $\rho(k)$ is the corresponding equilibrium density profile.

The case $E < 0$ could also be interpreted as the interface distribution of a fluid filling a half-space and subjected to a constant gravitational field.

(ii) Droplet Shapes under Glauber Dynamics in the Low-Temperature Limit

To each unit square of Z^2 we associate an Ising spin and consider the spin flip process ω_t in $\Omega = \{1, -1\}^{Z^2}$ with Glauber dynamics, in the presence of a constant magnetic field h. The generator of the process is defined by

$$(Gf)(\omega) = \sum_k c(k, \omega)[f(\omega_k) - f(\omega)]$$

where ω_k is the configuration obtained from ω by flipping the spin at k.[4] The speed functions $c(k, \omega)$ satisfy the detailed balance condition with respect to the Ising Hamiltonian $H(\omega) = -J \sum_{\langle k,l \rangle} \omega(k)\, \omega(l) - h \sum_k \omega(k)$ where the summation $\langle\ ,\ \rangle$ is on next-neighbor sites. A possible choice is (up to a factor)

$$c(k, \omega) \sim [1 - \tanh 2\beta J(2 - s(k, \omega))][1 - \alpha\omega(k)]$$

where $\alpha = \tanh \beta h$ and $s(k, \omega)$ is the number of neighboring spins to k with sign opposite to $\omega(k)$.

At very low temperature ($\beta \gg 1$), the speed functions for $s = 0, 1$ become negligibly small as compared to those for $s \geqslant 2$ which remain of the order of $(1 - \alpha\omega(k))$. We therefore introduce the limiting process with speed functions

$$c^*(k, \omega) = \begin{cases} 0 & s < 2 \\ \frac{1}{2}(1 - \alpha\omega(k)) & s = 2 \\ 1 - \alpha\omega(k) & s > 2 \end{cases}$$

letting $\beta \to \infty$, $h \to 0$ with $\alpha = \tanh(\beta h)$ fixed in $c(k, \omega)$.

Assume that the initial state μ_0 on Ω is concentrated on the configuration

$$\omega_0(k) = \begin{cases} 1 & k \in Z_+^2 \\ -1 & \text{otherwise} \end{cases}$$

Geometrically, ω_0 describes a $(+1)$ spin droplet filling the first quadrant of Z^2. Since the $s < 2$ events are forbidden, the droplet remains confined to the positive quadrant throughout its evolution and flips take place at its boundary only. Moreover, events with $s > 2$ will never occur when starting from ω_0. Hence all flips are $s = 2$ events and lead with probability $\frac{1}{2}(1 - \alpha)$ to a decrease and with probability $\frac{1}{2}(1 + \alpha)$ to an increase of the droplet area by one unit.

The case $\alpha < 0$ describes the unrestricted shrinkage of the $(+)$ spin region in a negative magnetic field. The expected area of erosion relative to the initial droplet area in ω_0 grows linearly with time.[2] In the case $\alpha > 0$ (positive field) we expect that the $(+)$ region, which remains confined to the positive quadrant, admits a unique asymptotic stationary distribution.

The main purpose of this note is to study this distribution and the corresponding typical shape of the droplet. This can be done as follows.

Any configuration ω which can be reached from ω_0 is characterized by the boundary which separates the two spin regions, i.e., by a sequence $\{B_k(\omega), k \in Z_+\}$ of nonnegative integer-valued random variables, nonincreasing in k, and the area of erosion of the $(+)$ region in configuration ω is $A(\omega) = \sum_{k=1}^{\infty} B_k(\omega)$ (cf. Fig. 2).

We can now define an isomorphism between the process ω_t governed by $c^*(k, \omega)$ with initial configuration ω_0 and the exclusion process η_t with initial configuration[1,2] η_0: To a $(+) \to (-)$ (resp., $(-) \to (+)$) spin flip corresponds a right (resp. left) move of a particle on Z with probability $p = \frac{1}{2}(1-\alpha)$ (resp., $q = \frac{1}{2}(1+\alpha)$); hence

$$r = \frac{p}{q} = \frac{1-\alpha}{1+\alpha} \tag{1.3}$$

It can then be verified inductively that the correspondence between ω and η is characterized by the relation

$$B_{k+1}(\omega) = k + \min\left\{n: \sum_{l>n} \eta(l) = k\right\} \tag{1.4}$$

We note that $\min_n\{\sum_{l>n} \eta(l) = k\}$ is the position in η of the particle which has k right neighbors. Since, by exclusion, particles do not cross, this particle already had k right neighbors initially, hence was located at $-k$ in η_0. We therefore conclude from (1.4) that the boundary variable $B_k(\omega)$ equals the total distance traveled by the particle originally located at $-k$, and that the area of erosion $A(\omega)$ equals the sum of distances traveled by all particles from their initial positions.

In Sec. 2, we derive the full distribution for the random variables $A(\omega)$ (Proposition 2.2) and $B_k(\omega)$ (Proposition 2.8). A brief discussion of the boundary process in its own right will also be given (Proposition 2.15). In Sec. 3, we introduce a scaling of the lattice spacing and the magnetic field h, which leads to a deterministic continuum limit of the model (Proposition 3.6).

From a physical viewpoint, it is important to remark that at low, but nonzero, temperature, the $s = 1$ processes have always a nonvanishing small probability. These processes, which are neglected in the speed functions $c^*(k, \omega)$, are responsible for the formation of "protuberances" which enable the droplet to grow beyond its initial quadrant. For small T one should therefore distinguish two time scales for the dynamics of a supercritical droplet, a slow time scale for the global growth of the droplet involving $s=1$ processes, and a fast time scale of the $s=2$ processes governing the instantaneous shape of the droplet. The results of the paper will describe adequately the typical instantaneous shape, as illustrated by the computer-generated picture of a droplet initially confined to a square, under low-temperature Glauber dynamics (cf. Fig. 1).

A description of droplet growth (in a positive field) requires taking the $s=1$ processes into account, in one way or another. If one wants to maintain a bijective map between the spin flip processes and the particles on a line, one has to relax the exclusion restriction. A simple approximation to the Glauber dynamics including the $s=1$ events corresponds to the process on the line where particles are allowed to pile up at any one site, but without crossing each other (in order to preserve uniqueness of the map without tagging the particles). Defining the correspondence between traveling distance and boundary coordinate as before, one checks that to each particle configuration now corresponds a one-valued (SOS) function

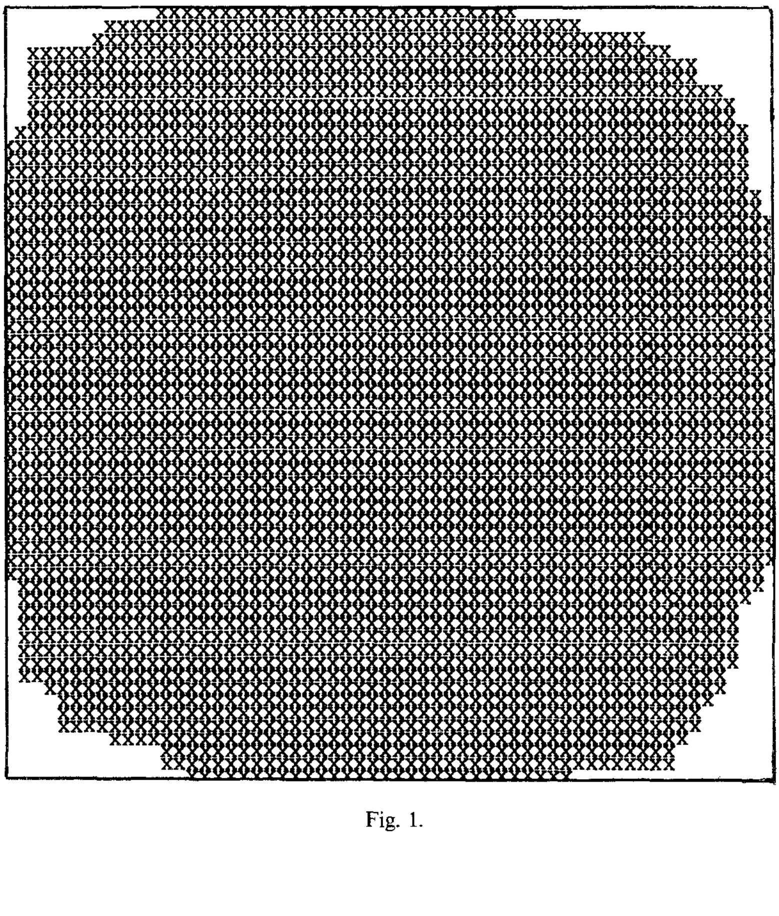

Fig. 1.

on Z_+ which, however, need not be monotonic and positive. The droplet can therefore grow into the right lower quadrant. This will be the subject of a further investigation.

2. BOUNDARY DISTRIBUTION

Let μ_∞ be the equilibrium measure on the configuration space Ω (corresponding to v_∞ on X).

Here and henceforth we introduce the notations

$$a(n) = \operatorname*{Prob}_{\mu_\infty}(A(\omega) = n), \quad b_k(n) = \operatorname*{Prob}_{\mu_\infty}(B_k(\omega) = n), \qquad n \in Z$$

for the equilibrium densities, and

$$\hat{a}(s) = \sum_{n=0}^{\infty} a(n)\, s^n, \quad \hat{b}_k(s) = \sum_{n=0}^{\infty} b_k(n)\, s^n, \qquad s \in \mathbb{R}_+$$

for the equilibrium generating functions of $A(\omega)$ and $B_k(\omega)$. Moreover, we use

$$b_{k_1 \cdots k_N}(n_1 \cdots n_N) = \operatorname*{Prob}_{\mu_\infty}(B_{k_1}(\omega) = n_1, \ldots, B_{k_N}(\omega) = n_N)$$

for the joint N-point density of $B_k(\omega)$, and define

$$\pi(m, n) = \prod_{l=m}^{n} (1 - r^l)$$

Lemma 2.1. At equilibrium, all configurations ω with erosion $A(\omega) = n$ have weight $a(0)\, r^n$.

Proof. In the exclusion picture, erosion from ω_0 corresponds to total displacement of particles from η_0. Let η, η' correspond to ω, ω' with erosions n and $n+1$ in such a way that η' can be reached from η in a single step. This step then necessarily consists in a particle moving from some $k \in Z$ to $k+1$. Let $A = \{l: \eta(l) = 1\}$, $B = \{l: \eta(l) = 0\}$, $k \in A$, $k+1 \in B$. Then, by (1.1) and (1.2),

$$v_\infty(\eta) = c \prod_A \rho(l) \prod_B (1 - \rho(l))$$

$$\begin{aligned} v_\infty(\eta') &= c \prod_{A \cup \{k+1\} - \{k\}} \rho(l) \prod_{B \cup \{k\} - \{k+1\}} (1 - \rho(l)) \\ &= v_\infty(\eta) \frac{\rho(k+1)}{1 - \rho(k+1)} \frac{1 - \rho(k)}{\rho(k)} = v_\infty(\eta)\, r \end{aligned}$$

Since $v_\infty(\eta_0) = \mu_\infty(\omega_0) = a(0)$, the result follows by induction. ∎

Proposition 2.2. At equilibrium, $A(\omega)$ has

density:	$a(n) = p_n r^n \pi(1, \infty)$
expectation:	$E(a) = \sum_{l=1}^{\infty} l r^l (1 - r^l)^{-1}$
variance:	$V(a) = \sum_{l=2}^{\infty} l^2 r^l (1 - r^l)^{-2}$

where p_n is the number of unrestricted partitions of the integer n.

Proof. The unique configuration with erosion $n = 0$ is ω_0, so

$$a(0) = \mu_\infty(\omega_0) = \nu_\infty(\eta_0)$$
$$= c \prod_{-\infty}^{0} \rho(l) \prod_{1}^{\infty} (1 - \rho(l)) = \frac{c}{2} (\pi(1, \infty))^2 \tag{2.3}$$

The number of contours with erosion n equals the number of nonincreasing sequences B_k with sum n which in turn equals p_n. By Lemma 2.1, each contour with erosion n contributes a weight $a(0)\, r^n$ so that

$$a(n) = p_n a(0)\, r^n \tag{2.4}$$

The generating function for unrestricted partitions is (Ref. 6, p. 111)

$$\hat{p}(s) = \sum_{0}^{\infty} p_l s^l = \prod_{1}^{\infty} (1 - s^l)^{-1}$$

Hence the generating function of $a(\omega)$ is

$$\hat{a}(s) = \sum_{n=0}^{\infty} p_n a(0)\, r^n s^n = a(0)\, \hat{p}(rs) \tag{2.5}$$

Since

$$1 = \sum_{0}^{\infty} a(n) = \hat{a}(s)|_{s=1} = a(0)\, \hat{p}(r)$$

we find

$$a(0) = \hat{p}(r)^{-1} = \prod_{1}^{\infty} (1 - r^l) = \pi(1, \infty) \tag{2.6}$$

and. from (2.4),

$$a(n) = p_n r^n \pi(1, \infty)$$

By differentiating the generating function (2.5), we also obtain

$$E(a) = \hat{a}'(s)|_{s=1} = \sum_{1}^{\infty} l r^l (1 - r^l)^{-1}$$

and

$$V(a) = (\hat{a}''(s) + \hat{a}'(s) - (\hat{a}'(s))^2)_{s=1} = \sum_{2}^{\infty} l^2 r^l (1 - r^l)^{-2} \quad ∎$$

As a side result we observe that the normalization constant c in (1.1) as obtained from (2.3) and (2.6) is

$$c = 2\pi^{-1}(1, \infty) \tag{2.7}$$

Proposition 2.8. At equilibrium, the boundary variables $B_k(\omega)$ have densities

$$b_k(n) = r^{kn} \pi(k, \infty)\, \pi^{-1}(1, n)$$

and joint N-point densities ($N \geqslant 2$)

$$\begin{aligned} & b_{k_1 \cdots k_N}(n_1 \cdots n_N) \\ & = r^{k_N n_N + \sum_{m=1}^{N-1} k_m(n_m - n_{m+1})} \times \pi(k, \infty)\, \pi^{-1}(1, n_N) \\ & \quad \times \prod_{m=1}^{N-1} [\pi(k_{m+1} - k_m, k_{m+1} - k_m + n_m - n_{m+1} - 1)\, \pi^{-1}(1, n_m - n_{m+1})] \end{aligned}$$

Proof. Let us consider in detail the case $N = 2$. The cases $N = 1$ and $N > 2$ can easily be established by analogy.

By Lemma 2.1 and formula (2.6), the weight of the boundary limiting the three regions I in Fig. 2 is

$$\pi(1, \infty)\, r^{k_2 n_2 + (k_1 - 1)(n_1 - n_2)}$$

Denote by $q_l^{(a)}$ the number of partitions of l into parts not exceeding a, and by $q_l^{(a,b)}$ the number of partitions of l into exactly b parts, none exceeding a. It is known (Ref. 6, pp. 111, 153) that the corresponding generating functions are

$$\hat{q}^{(a)}(r) = \pi^{-1}(1, a), \qquad \hat{q}^{(a,b)}(r) = \pi(a, a + b - 1)\, \pi^{-1}(1, b)\, r^b \tag{2.9}$$

We now observe that the numbers of admissible boundaries limiting an area l in regions II, III, and IV of Fig. 2 are, respectively,

$$q_l^{(k_1 - 1)}, \qquad q_l^{(n_2)}, \qquad q_l^{(k_2 - k_1, n_1 - n_2)}$$

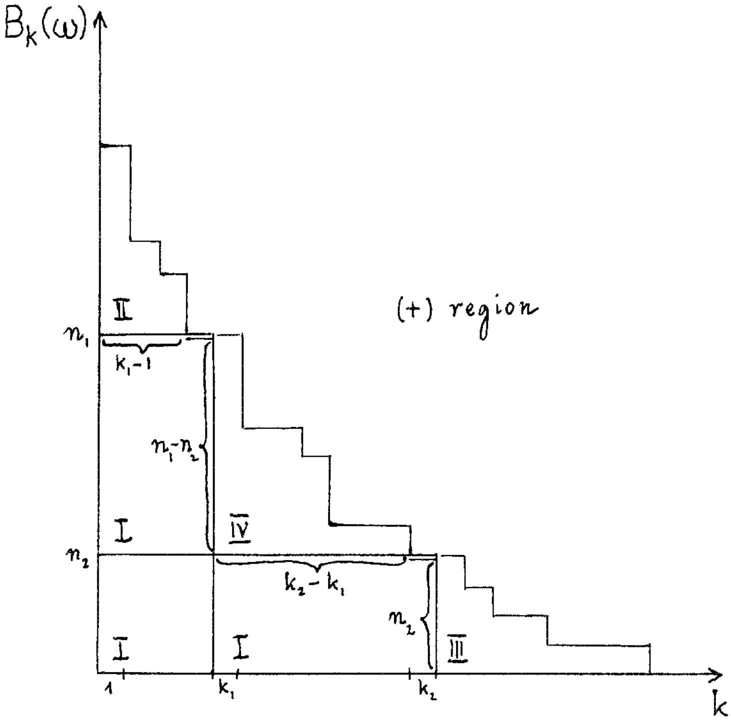

Fig. 2.

The total weight of all admissible boundaries is therefore

$$\pi(1, \infty)\, r^{k_2 n_2 + (k_1-1)(n_1-n_2)} \left(\sum_{l=0}^{\infty} q_l^{(k_1-1)} r^l \right) \left(\sum_{0}^{\infty} q_l^{(n_2)} r^l \right)$$

$$\times \left(\sum_{0}^{\infty} q_l^{(k_2-k_1, n_1-n_2)} r^l \right)$$

$$= \pi(1, \infty)\, r^{k_2 n_2 + (k_1-1)(n_1-n_2)} \hat{q}^{(k_1-1)}(r) \cdot \hat{q}^{(n_2)}(r) \cdot \hat{q}^{(k_2-k_1, n_1-n_2)}(r)$$

Inserting (2.9) we find

$$b_{k_1 k_2}(n_1 n_2) = r^{k_2 n_2 + k_1(n_1-n_2)} \pi(k_1, \infty)\, \pi^{-1}(1, n_2)$$

$$\times \pi(k_2 - k_1, k_2 - k_1 + n_1 - n_2 - 1)\, \pi^{-1}(1, n_1 - n_2)$$

This proves the proposition for $N=2$. ∎

Proposition 2.8 permits, at least in principle, a complete probabilistic analysis of the droplet shape at equilibrium. We conclude this section by a simple application which consists in viewing the sequence $\{B_k(\omega), k \in Z_+\}$ as a discrete "time" k random process in its own right.

Let $k_1 < k_2 < \cdots < k_{N-1} < k_N$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{N-1} \geqslant n_N$. The conditional probability in state μ^0_∞ that $B_{k_N}(\omega) = n_N$, given that $B_{k_i}(\omega) = n_{k_i}$, $i = 1,..., N-1$, is

$$P(k_N, n_N \mid k_{N-1}, n_{N-1},..., k_1, n_1) = \frac{b_{k_1 \cdots k_N}(n_1 \cdots n_N)}{b_{k_1 \cdots k_{N-1}}(n_1 \cdots n_{N-1})}$$

and the probability that $B_{k_N}(\omega) = n_N$ conditioned on $B_{k_{N-1}}(\omega) = n_{N-1}$ alone is

$$P(k_N, n_N \mid k_{N-1}, n_{N-1}) = \frac{b_{k_{N-1}}(n_{N-1} n_N)}{b_{k_{N-1}}(n_{N-1})} \tag{2.10}$$

A straightforward calculation shows that both expressions equal

$$r^{(k_N - k_{N-1}) n_N} (1 - r^N)^{-1} \, \pi(k_N - k_{N-1}, k_N - k_{N-1} + n_{N-1} - n_N - 1) \\ \times \pi(1, n_{N-1} - n_N) \tag{2.11}$$

This implies that the chain $\{B_k(\omega)\}$ is Markov.

Relations (2.10) and (2.11) also demonstrate that $P(k_2 n_2 \mid k_1 n_1)$ only depends on the difference $\Delta k = k_2 - k_1$ (and n_1, n_2). Hence the Markov chain is homogeneous in "time" k with general transition probabilities

$$\begin{aligned} P_{\Delta k}(n_2 \mid n_1) &\equiv P(k_1 + \Delta k, n_2 \mid k_1, n_1) \\ &= r^{\Delta k \cdot n_2} \pi(1, n_1) \, \pi^{-1}(1, n_2) \\ &\quad \times \pi(\Delta k, \Delta k + n_1 - n_2 - 1) \, \pi^{-1}(1, n_1 - n_2) \end{aligned} \tag{2.12}$$

The one-step transition matrix

$$P_1(n_2 \mid n_1) = r^{n_2} \pi(1, n_1) \, \pi^{-1}(1, n_2) \, \vartheta(n_1 - n_2) \tag{2.13}$$

(the Heaviside function ϑ has been added to include the case $n_1 < n_2$) admits

$$p(n) = \delta_{n0} \tag{2.14}$$

as the unique normalized right eigenvector. Geometrically, this means that the boundary must eventually converge to the x-axis for large k.

For $n_1 = n_2 = n > 0$, (2.12) reduces to

$$P_{\Delta k}(n \mid n) = r^{\Delta k \cdot n}$$

The likelihood of a horizontal plateau in the boundary thus decreases exponentially fast with its length Δk. Intuitively this indicates that large deviations from convexity of the boundary tend to be rare.

We summarize our results in the following:

Proposition 2.15. At equilibrium, the discrete boundary process $\{B_k(\omega)\}$ is a homogeneous Markov chain with (one step) transition matrix (2.13) and unique (universally attractive) stationary state (2.14).

3. CONTINUUM LIMIT

Consider the exclusion process on $Z_\Delta = \{n\Delta, n \in Z\}$ and the spin system on Z_Δ^2, with lattice spacing Δ. It is interesting to study the distribution of droplet shapes in the limit $\Delta \to 0$.

The typical linear size of an isolated $(+)$ droplet in a configuration of $(-)$ spins, in the presence of a small positive magnetic field h, is known to be of the order of h^{-1}.[5] To keep the physical size of the droplet invariant as $\Delta \to 0$, we let h converge to zero linearly with Δ. Since $\alpha = \tanh(\beta h)$ is proportional to h for small fields, the precise definition of the limit is

$$\Delta \to 0, \quad a \to 0; \qquad n\Delta = x \quad \text{and} \quad \frac{a}{\Delta} = \gamma > 0 \quad \text{fixed} \tag{3.1}$$

We begin with a heuristic derivation of the average stationary droplet shape in this limit. Equations (3.1) and (1.3) imply

$$(r_\Delta)^n = \left(\frac{1 - \Delta\gamma}{1 + \Delta\gamma}\right)^{x/\Delta} \to e^{-2\gamma x}, \qquad \Delta \to 0$$

and the density (1.2) converges to

$$\rho(x) = e^{-2\gamma x}(1 + e^{-2\gamma x})^{-1}$$

From (1.2) one infers that the expected asymptotic location u of a particle initially located at $-x$ satisfies

$$x = \int_u^\infty \rho(\xi)\, d\xi = \frac{1}{2\gamma} \log(1 + e^{-2\gamma u}) \tag{3.2}$$

since the expected final number of right neighbors equals the initial number x of right neighbors (by exclusion).

According to the general translation code between the exclusion and the low-temperature process, the expected boundary height y equals the distance $x+u$ traveled by the particle from $-x$ to u. Solving (3.2) for u yields

$$y(x)=x+u=-\frac{1}{2\gamma}\log(1-e^{-2\gamma x}) \tag{3.3}$$

or, in a form which exhibits the symmetry of the boundary curve relative to the 45° axis,

$$e^{-2\gamma x}+e^{-2\gamma y}=1$$

Next we show that, in the limit $\Delta \to 0$, the boundary distribution is concentrated on the curve (3.3). Let $b_k^\Delta(n)$ be the density of $B_k(\omega)$ parametrized by r_Δ, i.e., by Proposition 2.8:

$$b_k^\Delta(n)=(r_\Delta)^{kn}\,\pi_\Delta(k,\infty)\,\pi_\Delta^{-1}(1,n) \tag{3.4}$$

with $\pi_\Delta(m,n)\equiv\prod_{l=m}^{n}(1-r_\Delta^l)$. We show that the expectation functional

$$E_x^\Delta(g)=\sum_{n=0}^{\infty} g(n\Delta)\,b_k^\Delta(n), \qquad x=k\Delta \tag{3.5}$$

defined for suitable test functions g on the range Z_Δ of $B_k(\omega)$, converges weakly to the Dirac measure $\delta_{y(x)}(g)$.

Proposition 3.6. For any function $g\in D^\infty(0,\infty)$ (the space of infinitely differentiable functions with compact support in $[0,\infty)$):

$$\lim_{\Delta\to 0} E_x^\Delta(g)=g(y(x))$$

with $y(x)$ the expected boundary (3.3).

Proof. From the normalization relations $\sum_n b_k^\Delta(n)=1$ for the densities (3.4) follows

$$\begin{aligned}0&=\sum_n b_{k+l}^\Delta(n)-\sum_n b_k^\Delta(n)\\&=\pi_\Delta(k+l,\infty)\sum_n (r_\Delta^{(k+l)n}-r_\Delta^{kn})\,\pi^{-1}(1,n)\\&\quad+(\pi_\Delta(k+l,\infty)-\pi_\Delta(k,\infty))\sum_n r_\Delta^{kn}\pi_\Delta^{-1}(1,n)\end{aligned}$$

for any fixed $l \in Z_+$. Using the identity

$$\pi_\Delta(k+l, \infty) - \pi_\Delta(k, \infty)$$
$$= \pi_\Delta^{-1}(k, k+l-1)(1 - \pi_\Delta(k, k+l-1))\, \pi_\Delta(k, \infty)$$

and normalization once again, we obtain

$$0 = \pi_\Delta^{-1}(k, k+l-1) \left[\sum_n (r_\Delta^{ln} - 1)\, r_\Delta^{kn} \pi(k, \infty)\, \pi^{-1}(1, n) + 1 - \pi_\Delta(k, k+l-1) \right]$$
$$= \pi_\Delta^{-1}(k, k+l-1)[E_x^\Delta(r_\Delta^{ln}) - \pi_\Delta(k, k+l-1)]$$

For small Δ we set $r_\Delta^m \sim \exp(-2\gamma\, \Delta m)$ and $k\Delta = x$, $n\Delta = y$. Hence

$$E_x^\Delta(e^{-2\gamma y l}) = (1 - e^{-2\gamma x})(1 - e^{-2\gamma x(1+\Delta)}) \cdots (1 - e^{-2\gamma x(1+(l-1)\Delta)})$$
$$\sim (1 - e^{-2\gamma x})^l = \exp[l \log(1 - e^{-2\gamma x})] = e^{-2\gamma y(x) l}$$

The proposition is thus proved for the sequence $g_l(y) = \exp(-2\gamma y l)$ of exponentials, and the remaining argument rests on the Weierstrass approximation theorem according to which any $g \in D^\infty[0, \infty)$ can be uniformly approximated by linear combinations $\sum_{l=0} c_l g_l(y)$. (This can be verified by substituting $y = -(1/2\gamma) \log z$; $0 < z \leqslant 1$). ∎

We conclude this section by a direct argument showing that the densities $b_k^\Delta(n)$ are peaked at $y(x)$ in the limit $\Delta \to 0$.

For small Δ, (3.4) implies

$$\log b_k^\Delta(n) \sim kn \log(1 - 2\gamma\, \Delta) + \sum_{l=k}^{\infty} \log(1 - e^{-2\gamma \Delta l}) - \sum_{l=1}^{n} \log(1 - e^{-2\gamma \Delta l})$$
$$\sim \frac{1}{\Delta} G(x, y)$$

with

$$G(x, y) = -2\gamma xy + \int_x^\infty d\xi \log(1 - e^{-2\gamma\xi}) - \int_0^y d\xi \log(1 - e^{-2\gamma\xi})$$

Since by (3.3)

$$\frac{\partial}{\partial y} G(x, y_0) = -2\gamma x - \log(1 - e^{-2\gamma y_0})$$

vanishes if and only if $y_0 = y(x)$, the density $G(x, y)$ has a unique extremum in y for fixed x, and since

$$\sigma_0^{-1} \equiv -\frac{\partial^2}{\partial y^2} G(x, y_0) = 2\gamma e^{-2\gamma y_0}(1 - e^{-2\gamma y_0}) > 0$$

this extremum is a maximum and the normalized densities $b_k^\Delta(n)$ behave, for small Δ, as the Gaussians,

$$b_k^\Delta(n) \sim (2\pi\sigma_0 \Delta)^{-(1/2)} \exp\left(-\frac{(y - y_0)^2}{2\sigma_0 \Delta}\right)$$

This shows that the mean-square fluctuations of the boundary about $y(x)$ are of the order $\sigma_0 \Delta$ as $\Delta \to 0$.

It is also easily checked that the exponential generating function

$$\hat{a}^\Delta(s) = \sum_{n=0}^{\infty} a^\Delta(n) \exp(n \Delta^2 s)$$

for the area $n \Delta^2$ of erosion converges to

$$\exp\left[s \int_0^\infty y(x)\, dx\right]$$

and that the density $a^\Delta(n)$ is therefore concentrated on the area $\int_0^\infty y(x)\, dx$ in the limit $\Delta \to 0$.

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